JANUARY 2007 ANALYSIS QUALIFYING EXAM

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1. Problem 1

(a). Let x_n be a convergent sequence in $E \cdot F$. Then, $x_n = e_n f_n$ with $e_n \in E$ and $f_n \in F$. By Heine-Borel, E is compact so there exists a convergent subsequence $e_{n_k} \to e \in E$. Then, as $0 \notin E$, $\frac{x_{n_k}}{e_{n_k}} \in F$ is convergent sequence of elements in F.

As F is closed, $x/e \in F$, so that

$$x_n \to e \cdot \frac{x}{e} \in E \cdot F$$

so that $E \cdot F$ is closed.

(b). Note that $\mathbb{Z}$ and $\left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \cup \{0\}$ are both closed sets.

Their product, however, is $\mathbb{Q}$, which is not closed.

2. Problem 2

Let $N \in \mathbb{N}$. Consider $F_N(x) := \sum_{n=1}^N \frac{f(x+n)}{n}$. Then,

$$\begin{aligned} \int_{\mathbb{R}} F_N(x) dx &= \sum_{n=1}^N \int_{\mathbb{R}} \frac{f(x+n)}{n} dx \\ &= \sum_{n=1}^N \frac{1}{n} \cdot ||f||_1 \end{aligned}$$

Now, if $||f||_1 \neq 0$, then as $N \to \infty$, $F_N(x) \to \infty$, a contradiction to the definition of $F(x)$. Thus, $||f||_1 = 0$, whence $f(x) = 0$ a.e as contended.

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3. Problem 3

Note that

$$\begin{aligned}\int_{\mathbb{R}} e^x h(x)dx &= \int_{\mathbb{R}} e^y e^{x-y} \int_{\mathbb{R}} f(y)g(x-y)dydx \\ &= \int_{\mathbb{R}} e^{x-y} \int_{\mathbb{R}} e^y f(y)g(x-y)dydx\end{aligned}$$

Making the change of variable $u = x - y$, $du = dx$, and the above becomes:

$$\int_{\mathbb{R}} e^x g(x) \int_{\mathbb{R}} e^u f(u)dudx = \Big(\int_{\mathbb{R}} e^x g(x)dx\Big)\Big(\int_{\mathbb{R}} e^x f(x)dx\Big)$$

4. Problem 4

$(a) \implies (b)$: Assume f is absolutely continuous with $f(0) = 0$. Then, set $A := \big(f'\big)^{-1}(\{1\})$. This is measurable as f' is measurable. Then,

$$\begin{aligned}f(x) &= \int_0^x f'(t)dt \\ &= \int_0^1 f'(t)\chi_{[0,x]}(t)dt \\ &= \int_A \chi_{[0,x]}(t)dt + \int_{A^c} 0 \cdot \chi_{[0,x]}(t)dt \\ &= \int_0^1 \chi_{[0,x]\cap A}(t)dt \\ &= m\big(A \cap [0,x]\big)\end{aligned}$$

$(b) \implies (a)$: Note that $f(0) = m(\{0\})$ $m(\varnothing)$ are both 0, so $f(0) = 0$. Also, note that

$$f(x) = \int_0^x \chi_A(t)dt$$

Whence f is absolutely continuous, so that $f(x) = \int_0^x f'(t)dt$, and we see

$$\int_0^1 f'(t) - \chi_A(t)dt = 0$$
$$\implies f'(t) = \chi_A(t) \text{ a.e}$$
$$\implies f'(x) \in \{0,1\} \text{ a.e}$$

Which yields the result.

5. Problem 5

(a). Note that by the existence of $f'(0)$, Taylor's theorem guarantees that $f(x) = xf'(0) + xh_1(x)$ where $h_1(x) \to 0$ as $x \to 0$. Then, we see

$$\begin{aligned}\int_0^1 x^{-3p/2}|g(x)|^p dx &= \int_0^1 x^{-p/2}|f'(0) + h_1(x)|^p dx \\ &\leqslant 2^p \int_0^1 x^{-p/2}\Big(|f'(0)|^p + |h_1(x)|^p\Big)dx \\ &\leqslant 2^p\Big(|f'(0)|^p + ||h_1(x)||_\infty^p\Big)\int_0^1 x^{-p/2}dx \\ &< \infty\end{aligned}$$

Where $\int_0^1 x^{-p/2}dx < \infty$ since $p \in [1,2)$.

(b). Consider $f(x) := x^{1/2}$. This is certainly continuous, but $f'(0)$ does not exist. Also,

$$\int_0^1 |g(x)|^p dx = \int_0^1 \frac{1}{x^p}dx = \infty$$

6. Problem 6

Note that outer measure always exists. We can find open sets E_n such that

$$m^*(E_n \backslash A) < \frac{1}{n}$$

Set $E := \bigcap_{n=1}^\infty E_n$; certainly $m^*(E\backslash A) = 0$ by selection, and E is measurable as the countable intersection of open sets.

Now, if $F \subset A$, we have that $F^c \supset A^c$ in which case

$$m^*(E \backslash F) \leqslant m^*(E \backslash A) = 0$$

Since E and F are measurable, $m^* = m$, so that

$$m(E \backslash F) = 0$$

7. Problem 7

We argue by contraposition. Assume that f has no zeroes; then $1/f$ is holomorphic and by the maximum modulus principle,

$$\frac{1}{|f|} \leqslant \frac{1}{M}$$

Rearranging, we then see

$$M \leqslant |f| \leqslant M$$

in which case $|f|$ is constant, so that f must be constant.

8. Problem 8

We work with the standard Wirtinger derivatives for convenience. Assume that $\overline{f}g$ is holomorphic, so that

$$\frac{\partial}{\partial \overline{z}}(\overline{f}g) = 0$$

$$\implies \overline{\frac{\partial f}{\partial z}} \cdot g + f \frac{\partial g}{\partial \overline{z}} = 0$$

In which case, since g is holomorphic, we have $\overline{\frac{\partial f}{\partial z}} \cdot g = 0$; then either $g = 0$ or $\frac{\partial f}{\partial z} = 0$. If $\frac{\partial f}{\partial z} = 0$, holomorphicity of f already implies that $\frac{\partial f}{\partial \overline{z}} = 0$, in which case we deduce that f is constant.

If $\frac{\partial f}{\partial z} \neq 0$, then we deduce that $g = 0$, as contended.

9. Problem 9

(a). False. Consider $\sin(1/x)$. This is certainly bounded and continuous on $(0, 1)$ just by definition. Let $\delta > 0$; we can find $N \in \mathbb{N}$ such that

$$\frac{1}{\pi n} - \frac{1}{\pi n + \frac{\pi}{2}} < \delta$$

for all $n > N$. However, we see that

$$|\sin(n\pi) - \sin(n\pi + \pi/2)| = 1$$

So this is not uniformly continuous.

(b). True. By homogeneity, we may assume without loss of generality that $||f||_2 = ||g||_2 = 1$. Then, we see

$$\begin{aligned} \int_0^1 fg dx &= \int_0^1 f(g-1)dx \\ &\leqslant ||g-1||_2^2 \cdot ||f||_2^2 \\ &= 2 - 2||g||_1 \end{aligned}$$

Now, consider the transformations

$$f \mapsto c \cdot f, \quad g \mapsto \frac{g}{c}$$

for $c \neq 0$. Then, the lefthand side of the above string of equalities remains unchanged, and we are left with

$$||fg||_1^2 \leqslant c^2 - 2c||g||_2 + 1$$

Optimizing in c (that is, just take the derivative wth respect to c and set equal to 0), we see the minimum is obtained for $c = ||g||_1$, so that

$$||fg||_1^2 \leqslant 1 - ||g||_1^2$$

Which was to be proved.

(c). False. Consider $f_n(x) := \sin(n^2 x)$. One easily sees that $||f_n||_1 \leqslant \frac{1}{n^2}$, but $f_n(x) \not\to 0$.

(d). True. Argue by contraposition; if f has no pole at a, then in some neighborhood of a, we may write

$$f(z) = \sum_{n \geqslant 0} a_n (z-a)^n$$

Then,

$$f'(z) = \sum_{n \geqslant 1} n a_n (z-a)^{n-1}$$

And this also has no pole at a.

(e). True. Note that

$$2^p(|f_n|^p + |f|^p) - |f_n 0 f|^p \geqslant 0$$

So that upon taking norms and employing Fatou's lemma,

$$2^{p+1}||f||_p^p \leqslant \liminf_{n\to\infty} \left(2^p(||f_n||_p^p + ||f||_p^p) - ||f - f_n||_p^p\right)$$

Which implies that

$$\limsup_{n\to\infty} ||f_n - f||_p^p = 0$$

and we get the result.